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On a Parametric Spline function

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***Abstract** :This paper is concerned with the development of non-polynomial spline function approximation method to obtain numerical solution of ordinary and partial differential equations. The parametric spline function which depends on a parameter $p < 0$, is discussed which reduced to the ordinary cubic spline [1] when the parameter $p = 0$.*
The numerical method is tested by considering an example.
***Keywords :** Cubic spline function, Parametric spline function, finite difference method.*

I. Introduction

We consider a mesh Δ with nodal points x_i on the interval $[a,b]$ such that $\Delta : a = x_0 < x_1 < ... < x_N = b,$ where $h = x_i - x_{i-1}, i = 1(1)N$. Assume we are given the values $\{y_i\}_{i=0}^{N}$ of the function $y(x)$,with $[a,b]$ as its domain of definition. A spline function of degree ***m*** with nodes at the points $x_i, i = 1,.., N$ is a function $s_\Delta(x)$ with the following properties :

(i) $s_\Delta(x)$ is a polynomial of degree m in each subinterval $[x_i, x_{i+1}], i = 0,1,2,..., N-1$.

(ii) $s_\Delta(x)$ and its first $(m-1)$ derivatives are continuous on $[a,b]$.

A cubic spline function $s_\Delta(x)$, of class $C^2[a,b]$ interpolating to a function $y(x)$ defined on $[a,b]$ is such that in each interval $[x_{i-1}, x_i]$, $s_\Delta(x)$ is a polynomial of degree at most three and the first and second derivatives of $s_\Delta(x)$ are continuous on $[a,b]$.

II. Parametric Spline Function.

Given an interval $[a,b]$ and a mesh points with knots $a = x_0 < x_1 < ... < x_n = b,$ with $h = x_i - x_{i-1},\ i = 1,2,..., N.$ A function $s_i(x) \subset C^2[a,b]$ which interpolates the function $\boldsymbol{y(x)}$ at the knots $\boldsymbol{x_i}$ depends on the parameter $p < 0$ and reduces to a cubic spline function in the interval $[x_{i-1}, x_i]$ as $p = 0$ is termed a parametric spline function. The parametric spline function when $p > 0$ is discussed in [2]. If $s_i(x)$ is a parametric spline function in the interval $[x_{i-1}, x_i]$, then it satisfies the following differential equation:

$$s_i''(x) - p^2 s_i(x) = \left(M_{i-1} - p^2 y_{i-1}\right)\left(\frac{x_i - x}{h}\right) + \left(M_i - p^2 y_i\right)\left(\frac{x - x_{i-1}}{h}\right) \qquad (1)$$

where $M_i = y''(x_i), s_i(x_i) = y(x_i)$ and $\boldsymbol{p}$ is a parameter and we denote to $y(x_i)$ by y_i,

Solving the differential equation (1) on the interval $[x_{i-1}, x_i]$, subject to $s_i(x_i) = y_i$ and $s_{i-1}(x_{i-1}) = y_{i-1}$ we obtain:

$$s_i(x) = \frac{h^2}{k^2 \sinh k}\{M_i \sinh kz_{i-1} - M_{i-1}\sinh kz_i\}$$
$$-\frac{h^2}{k^2}\{(M_i - w y_i)z_{i-1} - (M_{i-1} - w y_{i-1})z_i\} \qquad (2)$$

where $z_{i-1} = (\frac{x - x_{i-1}}{h})$, $w = \frac{k^2}{h^2}$ and $k = ph$

The continuity of the first derivative of $s_i(x)$ at $\boldsymbol{x_i}$ in the form $s'(x_i) = s'_{i+1}(x_i)$ which gives

$$y_{i+1} - 2y_i + y_{i-1} = h^2\left\{\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1}\right\} \tag{3}$$

where

$$\alpha = k^{-2}\left(1 - k\operatorname{csch} k\right) \tag{4}$$

$$\beta = -k^{-2}\left(1 - k\coth k\right) \tag{5}$$

The consistency relation for (3) leads to equation $2\alpha + 2\beta = 1$. Which may also be expressed as $k/2 = \tan k/2$. This equation has a zero root and an infinite number of non-zero roots. The smallest positive being $k = 8.986818916$ and for $k/2 = \tan k/2 \neq 0$, $\alpha + \beta = 1/2$. For the cubic spline $\alpha = 1/6,\ \beta = 1/3$.

From equation (2) some spline relations are derived which useful in solving boundary value problems . differentiate (2) at x_i , x_{i+1} then

$$s'_i(x_i) = -h\left(\alpha M_{i+1} + \beta M_i\right) + \left(\frac{y_{i+1} - y_i}{h}\right) \tag{6}$$

$$s'_i(x_{i+1}) = h\left(\beta M_{i+1} + \alpha M_i\right) + \left(\frac{y_{i+1} - y_i}{h}\right) \tag{7}$$

$$s'_i(x_i) + s'_i(x_{i+1}) = h\left(\beta - \alpha\right)\left(M_{i+1} + M_i\right) + 2\left(\frac{y_{i+1} - y_i}{h}\right) \tag{8}$$

$$s'_i(x_{i+1}) + s'_i(x_i) = h\left(\beta + \alpha\right)\left(M_{i+1} + M_i\right) \tag{9}$$

when $p = 0$ equation (1) take the form

$$s''_i(x) = \left(M_{i-1}\right)\left(\frac{x_i - x}{h}\right) + \left(M_i\right)\left(\frac{x - x_{i-1}}{h}\right) \tag{10}$$

which leads to the cubic spline function

$$\begin{aligned} s_i(x) &= \left(M_{i-1}\right)\frac{(x_i - x)^3}{6h} + \left(M_i\right)\frac{(x - x_{i-1})^3}{h} \\ &\quad + \left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right)\frac{(x_i - x)}{h} + \left(y_i - \frac{h^2}{6}M_i\right)\frac{(x - x_{i-1})}{h} \end{aligned} \tag{11}$$

$x_{i-1} \le x \le x_i$.

III. Application

(a) Numerical method for solving second-order differential equation.

Consider the second order differential equation

$$y'' = f(x, y), \qquad a \le x \le b \tag{12}$$

$$y(a) = y_0 \tag{13}$$

$$y(b) = y_N \tag{14}$$

The difference equation (3) can be used to determine the approximate values of $y(x_i)$ at the knots points $\{x_i\}, i = 1, 2, \ldots, N$ where $N = \dfrac{b-a}{h}$. The difference equation when equivalent to (3) is given by

$$y_{i+1} - 2y_i + y_{i-1} = \frac{h^2}{k^2}\left\{(1 - k\ \operatorname{csch} k) f_{i+1} - 2(1 - k\coth k) f_i + (1 - k\operatorname{csch} k) f_{i-1}\right\} \tag{15}$$

where $f_i = f(x_i, x_i)$

Equation (15) is explicit in y_{i+1} and its suitable for solving the differential equation (12)-(14).

(b) Numerical Example.

Consider the differential equation which describe the fluid flow inside a circular cylinder in the polar form

$\nabla^2\psi = 0$

where $\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$

with boundary conditions

$\psi = 0,$ on $r = 1$

$\psi = r\sin\theta$ as $r \to \infty$

$\psi = 0$ for $\theta = 0, \pi$

By using the transformation $r = e^t$ the problem transform to

$$\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial\theta^2} = 0 \qquad (16)$$

with boundary conditions

$\psi = 0,$ on $t = 0$

$\psi = e^t\sin\theta$ as $r \to \infty$

$\psi = 0$ for $\theta = 0, \pi$

by considering the parametric spline function approximation in $t-$direction with step size $h = 0.2$ and mish points $t_i = t_0 + ih,\ i = 1, 2, ..., N$ In θ- direction we apply finite difference approach with step size $l = 0.1\pi$ with knots points $\theta_j = \theta_0 + jl,\ \ j = 1, 2, ..., L$ .and t_∞ is taken as 0.3 . Equation (16) can be written in the form

$$M_{i,j} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{l^2} = 0$$

and by using equation (3) we have the system

$$\psi_{i,j} = \frac{1}{2}(\psi_{i+1,j} + \psi_{i-1,j}) - \frac{h^2}{2}\left\{\alpha M_{i+1,j} + 2\beta M_{i,j} + \alpha M_{i-1,j}\right\}$$

$$M_{i,j} = -\frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{l^2}$$

$$i = 1, 2, ..., N-1, \quad j = 1, 2, ..., L-1.$$

From the boundary conditions we have

$M_{0,j} = 0,$

$M_{N,j} = \frac{1}{l^2}\left(\psi_{N,j+1} - 2\psi_{N,j} + \psi_{N,j-1}\right),$

$\psi_{0,j} = 0,$

$\psi_{N,j} = e^3\sin\theta_j$

The exact solution $\psi = 2\sinh t\sin\theta$ we use Mathematica program to obtain the following numerical result with $N = 6, L = 10$. The computational results are present in the following table with the exact values between the brackets. This problem has earlier been discussed in [2].

	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
$\theta = 0.1\pi$	0.125356 (0.124432)	0.255717 (0.253859)	0.39629 (0.393474)	0.552692 (0.54888)	0.731165 (0.726314)
$\theta = 0.2\pi$	0.23844 (0.236686)	0.486403 (0.485633)	0.753791 (0.748431)	1.5128 (1.4403)	1.39067 (1.38153)
$\theta = 0.3\pi$	0.328185 (0.325768)	0.669477 (0.664611)	1.0375 (1.03013)	1.44697 (1.43699)	1.91421 (1.90152)

$\theta = 0.4\pi$	0.385805 (0.382964)	0.787017 (0.781297)	1.21966 (1.21099)	1.70101 (1.68928)	2.25029 (2.23537)
$\theta = 0.5\pi$	0.405659 (0.402627)	0.827519 (0.821505)	1.28243 (1.27331)	1.78855 (1.77621)	2.3661 (2.3504)
$\theta = 0.6\pi$	0.385805 (0.382964)	0.787017 (0.781297)	1.21966 (1.21099)	1.70101 (1.68928)	2.25029 (2.23537)
$\theta = 0.7\pi$	0.328185 (0.325768)	0.669477 (0.664611)	1.0375 (1.03013)	1.44697 (1.43699)	1.91421 (1.90152)
$\theta = 0.8\pi$	0.23844 (0.236686)	0.486403 (0.485633)	0.753791 (0.748431)	1.5128 (1.4403)	1.39067 (1.38153)
$\theta = 0.9\pi$	0.125356 (0.124432)	0.255717 (0.253859)	0.39629 (0.393474)	0.552692 (0.54888)	0.731165 (0.726314)

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